Highly-Smooth Zero-th Order Stochastic Online Optimization

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Motivations 1/2

- Final step in learning, statistics... minimization of loss $f(x)$

Structural assumption: Convex optimization

- Regularity of loss may vary
  - "Non-smooth", with "kinks" (hinge loss)
  - "smooth", if bounded second derivative
  - $f(x) = \log(1 + \exp(a^\top x))$, $f(x) = \|x - x^\star\|_H^2$, much smoother

Regularity assumption: High smoothness

2nd, 3rd, ... derivatives bounded

- Noise in the data (i.i.d.) or in the program

Noise assumption: Stochastic optimization

$f(\hat{\theta}) = \mathbb{E}\|\theta - \hat{\theta}\|^2$, get to observe/compute $\|\theta - \hat{\theta}\|^2$

Motivations 1/2

- Computational potential
  - Closed-form for f, can compute $\nabla f(x)$
  - No-closed form; $f(x)$ is (noisy) output of lengthy simulation. Cannot compute $\nabla f(x)$, just $f(x)$

Computational assumption: 0-th order

  - (French nuclear agency) Optimization of “shape of cores" to minimize the maximal temperature during accident (48h). Complex thermodynamic system, simulations last 1 day.

- Online vs Offline Optimization
  - Off-line: loss function f does not change
  - On-line: sequence of loss functions f_n

Stationary assumption. Offline **and** Online

- Practical (non-asymptotic), explicit dependency in dimension

Objectives 1

Stochastic Optimization of a convex function

– Unknown Mapping $f : \mathbb{R}^d \to \mathbb{R}$, constraint set $\mathcal{X} \subset \mathbb{R}^d$

i) Query $x_1 \in \mathbb{R}^d$, feedback $f(x_1) + \xi_1$ (0-th order), $\xi_1 \sim \mathcal{N}(0, \sigma^2)$

(or $\nabla f(x_1) + \vec{\xi_1}$ for 1st order, $\nabla^2 f(x_1) + \Xi_1$, 2nd order)

i bis) Output $x_1^\star \in \mathcal{X}$ the guessed "minimum" of f

ii) Query $x_2 \in \mathbb{R}^d$, get $f(x_2) + \xi_2$, output $x_2^\star \in \mathcal{X}$, etc.

Performance of an algorithm after T steps: $f(x_{T+1}^\star) - f^\star$
with $f^\star = \min_{x \in \mathcal{X}} f(x) = f(x^\star)$

Objectives 2

Stochastic Online Optimization of convex functions

– **Sequence** of mappings $f_t : \mathbb{R}^d \to \mathbb{R}$, constraint set $\mathcal{X} \subset \mathbb{R}^d$

i) Query $x_1 \in \mathbb{R}^d$, feedback $f_1(x_1) + \xi_1$ (0-th order), $\xi_1 \sim \mathcal{N}(0, \sigma^2)$
(or $\nabla f_1(x_1) + \vec{\xi_1}$ for 1st order, $\nabla^2 2 f_1(x_1) + \Xi_1$, 2nd order)

i bis) Output $x_2^\star \in \mathcal{X}$ the guessed "minimum" of f_2

ii) Query $x_2 \in \mathbb{R}^d$, get $f_2(x_2) + \xi_2$, output $x_3^\star \in \mathcal{X}$, etc.

Performance of an algorithm after T steps, "Regret"

$$\frac{1}{T}\sum_{t=1}^{T} f_t(x_t^*) - \min_{x^\star \in \mathcal{X}} \frac{1}{T}\sum_{t=1}^{T} f_t(x^\star)$$

Not bandit ! $x_k^\star \neq x_k$ and $x_k \notin \mathcal{X}$

Assumptions: strong cvxity & smoothness

Strongly convex: Intuitively, $f''(x) \geq \mu$ or $\nabla^2(f)(x) \succeq \mu Id$

f is μ-strongly convex iff

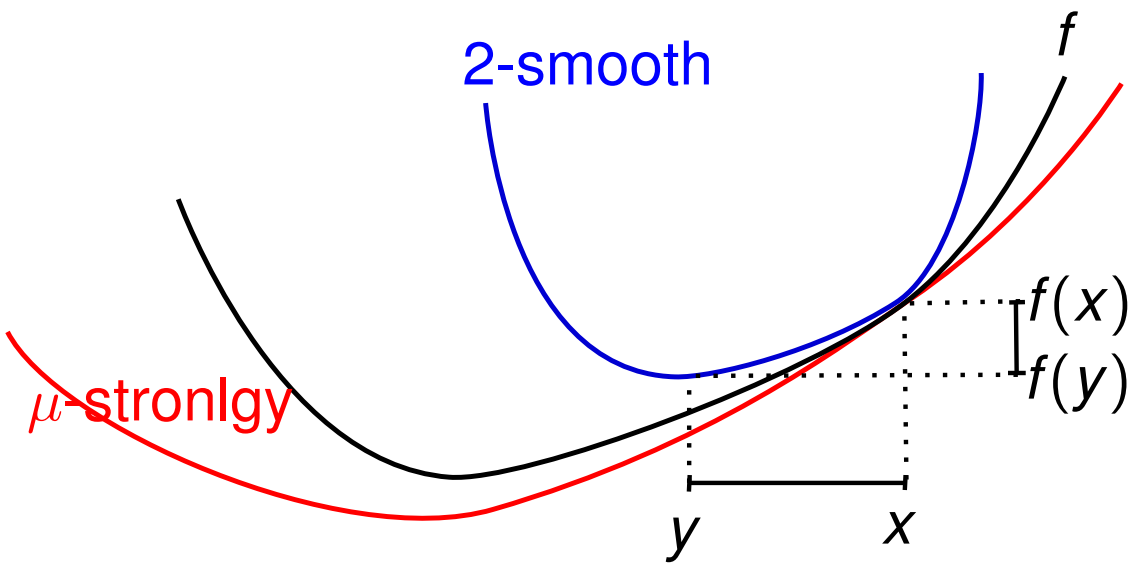
$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{\mu}{2}\|x - y\|^2$$

- Any convex mapping is 0-strongly convex
- f has "no flat part" (linear or 3rd order)

f is 2-smooth iff

$$f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{M_2^2}{2}\|x - y\|^2$$

Picture strong cvxity & smoothness



strong cvx quadratic **lower-bound**

smooth quadratic **upper-bound**

High β-smoothness

2-smooth $\left|f(y) - \text{Taylor}_1[f](x)\right| \leq \frac{M_2^2}{2!}\|y-x\|^2$

β-smooth $\left|f(y) - \text{Taylor}_{\beta-1}[f](x)\right| \leq \frac{M_\beta^\beta}{\beta!}\|y-x\|^\beta$

f is β-smooth iff

$$\left|f(y) - \sum_{|m|\leq\beta-1} \frac{1}{m!} f^{(m)}(x)(y-x)^m\right| \leq \frac{M_\beta^\beta}{\beta!}\|y-x\|^\beta$$

with $f^{(m)}(x)(y-x)^m = \frac{\partial^{m_1+\ldots+m_d}}{\partial^{m_1}\ldots\partial^{m_d}} f(x)(y_1-x_1)^{m_1}\ldots(y_d-x_d)^{m_d}$

On the high-regularity

$$\left| f(y) - \sum_{|m| \leq \beta - 1} \frac{1}{m!} f^{(m)}(x)(y-x)^m \right| \leq \frac{M_\beta^\beta}{\beta!} \|y - x\|^\beta$$

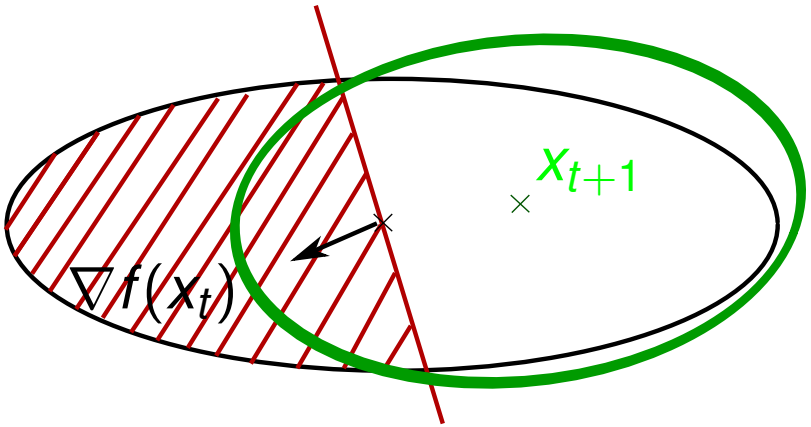
- 0-smooth = bounded (by M_0 if $\mathcal{X}$ compact)
- 1-smooth = M_1-Lipschitz

Lemma

If f is both β_1- and β_2-smooth, then f is β-smooth for all $\beta \in [\beta_1, \beta_2]$

- $a_\beta M_\beta^\beta \leq 2(\alpha_{\beta_1} M_{\beta_1}^{\beta_1})^{\frac{\beta_2 - \beta}{\beta_2 - \beta_1}} (\alpha_{\beta_2} M_{\beta_2}^{\beta_2})^{\frac{\beta - \beta_1}{\beta_2 - \beta_1}}$, and α_k fct only of k
- Logistic Regression $f(x) = \mathbb{E}_a \log(1 + \exp(-a^\top x))$ for a random. If $\|a\| \leq R$, then f is ∞-smooth and $M_\beta \leq \beta R$

Optim without noise: Ellipsoid method



- Decrease the volums of Ellipsoids (by a constant factor $\exp(-\frac{1}{2d})$)
- Exponential decay [Yudin & Nemirovski]

$$\min_t f(x_t) - f^\star \leq O\Big(M_0 R \exp(-\frac{1}{2}\frac{T}{d^2})\Big)$$

Gradient methods

$$x_{t+1} = \Pi_{\mathcal{X}}\Big(x_t - \eta \nabla f(x_t)\Big)$$

1-smooth $f(\overline{x}_T) - f^\star \leq \frac{1}{\sqrt{T}} R M_1$, with $\eta = \frac{\|\mathcal{X}\|}{M_0\sqrt{T}}$

2-smooth $f(x_T) - f^\star \leq \frac{1}{T}\|x_1 - x^\star\|^2 M_2^2$, with $\eta = \frac{1}{M_2^2}$

with acceleration $f(x_T) - f^\star \leq \frac{1}{T^2}\|x_1 - x^\star\|^2 M_2^2$

1-smth + strg $f(\hat{x}_T) - f^\star \leq \frac{1}{\mu T} 2M_1^2$, with $\eta = \frac{2}{\mu(t+1)}$

2-smth + strg $f(x_T) - f^\star \leq \left(\frac{\kappa-1}{\kappa+1}\right)^{2(T-1))} M_2^2 \|x_1 - x^*\|^2$,

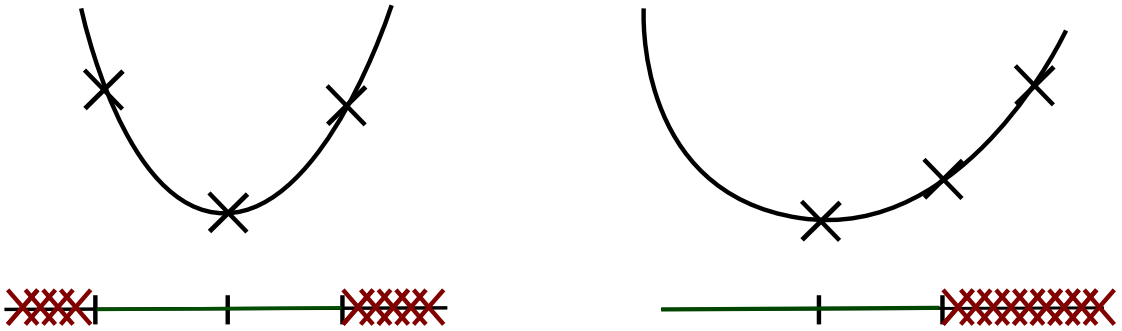
where $\kappa = M_2^2/\mu$ is the condition number

0th-order methods

– Using $2d$ queries to get $\nabla f(\cdot)$, same rate except $t \rightsquigarrow \frac{t}{2d}$

$$\frac{f(x+\delta\vec{u}) - f(x-\delta\vec{u})}{2\delta} \simeq \nabla f(x)^\top \vec{u}$$

– True 0-th order method ("Pyramids" in higher dimensions)



Convergence: $O\left(d \exp(-\frac{T}{d^7})\right)$ [Nemirovski-Yudin]

From 1st to 0th order: **Rates multiplied by d**

With noise, $\nabla f(x_t) + \vec{\xi}_t$

- Cutting algo (ellipsoid, pyramids...).
  Query d times the same points (lower the variance)
- Stochastic gradient

$$x_{t+1} = x_t - \eta\Big(\nabla f(x_t) + \vec{\xi}_t\Big),\ \mathbb{E}\|\xi_t\|^2 \simeq d\sigma^2$$

non-strongly rates $O\Big(\Big(\frac{1}{\sqrt{T}}(M_1 R + \sqrt{d}\sigma)\Big) = O\Big(\sqrt{\frac{d}{T}}\Big)$

μ-strongly rates $O\Big(\frac{1}{\mu T}(M_1^2 + d\sigma^2)\Big) = O\Big(\frac{d}{\mu T}\Big)$

Without noise to noisy: **Lose a factor d**

[Shamir, Tang], [Hazan et al], [Bach, Moulines], [Bach et al]...

Sum up - Objectives

Principal rates (without smoothness for the 1st column)

	1st	noisy	0th	0/noise	Our results
cvx	$\sqrt{\frac{1}{T}}$	$\sqrt{\frac{d}{T}}$	$\sqrt{\frac{d}{T}}$	$\sqrt{\frac{d^2}{T}}$?	$\left(\sqrt{\frac{d^2}{T}}\right)^{\frac{\beta-1}{\beta}}$ & $\left(\sqrt{\frac{d^2}{T}}\right)^{\frac{\beta}{\beta+1}}$, $\beta \leq 2$
μ-strg	$\frac{1}{\mu T}$	$\frac{d}{\mu T}$	$\frac{d}{\mu T}$	$\frac{d^2}{\mu T}$?	$\left(\frac{d^2}{\mu T}\right)^{\frac{\beta}{\beta+2}}$ or $\frac{1}{\mu^2}\left(\frac{d^2}{T}\right)^{\frac{\beta+1}{\beta}}$

Existing results

- Minimax speed (not strg) $\frac{\mathrm{poly}(d)}{\sqrt{T}}$ [Bubeck-Eldan],[Rakhlin et al.]
- Strongly AND smooth, algo $\sqrt{\frac{d^2}{\mu T}}$[Hazan, Levy]
- Strongly OR smooth, algo $T^{-1/3}$ [Agarwal et al][Saha,Tewari])
- Only Convex $T^{-1/4}$ [Flaxman et al][Kleinberg]
- Strongly and β-smooth, $T^{-\frac{\beta+1}{\beta}}$ [Polyak,Tsybakov]

The tricks 1/2: Stoch. Gradient

- Build an estimate of $f'(x)$ based on the values of $f(x) + \xi$

$f'(x) \simeq \frac{f(x+\delta)-f(x-\delta)}{2\delta} = f'_\delta(x)$, $f_\delta(x) := \frac{1}{2}\int_{-1}^{1} f(x+\delta v)dv = \mathbb{E}_{|v|\leq 1} f(x+\delta v)$

  - Draw $\varepsilon = \pm 1$ with proba $1/2$, $g(z) = \frac{f(x+\varepsilon\delta)\varepsilon}{\delta}$

    Unbiased: $\mathbb{E}_\varepsilon[g(z)] = f'_\delta(x)$, $\mathbb{E}_\varepsilon[g^2(z)] \leq \frac{1}{\delta^2}$

- Stochastic gradient descent (w.r.t. f_δ which **is convex**):

  $f_\delta(x_T^\star) - f_\delta(x_\delta^\star) \lesssim \frac{1}{\delta\sqrt{T}}$, so $f(x_T^\star) - f(x^\star) \lesssim \frac{1}{\delta\sqrt{T}} + \delta M_1 \lesssim \frac{1}{T^{1/4}}$

The tricks 1/2: Stoch. Gradient

- In higher dimension

$$\mathbb{E}_{\|u\|=1} \frac{d}{\delta} f(x+\delta u) u = \nabla f_\delta(x) \text{ with } f_\delta(x) = \mathbb{E}_{\|v\|\leq 1} f(x+\delta v)$$

  - $\mathbb{E}\|g(\xi)\|^2 \leq \frac{d^2}{\delta^2}$ and $\left| f_\delta(x) - f(x) \right| \leq M_1 \delta$

$$f(x_T^\star) - f^\star \lesssim \frac{d}{\delta\sqrt{T}} + \delta \lesssim \left(\frac{d^2}{T}\right)^{1/4}$$ [Nemirovski-Yudin, Flaxman et al., Hazan et al.]

The tricks 2/2: Kernels

- β-regularity on f (in 1 dimension)

$$\left| f(x+r) - \sum_{m=0}^{\beta-1} \frac{r^m}{m!} f^{(m)}(x) \right| \leq \frac{M_\beta^\beta}{\beta!} r^\beta$$

- $k(\cdot) : [-1,1] \to \mathbb{R}$ such that
  - $\int_{-1}^{1} rk(r)dr = 1$
  - $\int_{-1}^{1} r^m k(r)dr = 0$ for all $m \in \{0, 2, \ldots, \beta\}$

$$\left| \int_{-1}^{1} f(x+r\delta)rk(r)dr - f(x) \right| \leq \frac{\delta^\beta M_\beta^\beta}{\beta!} \int_{-1}^{1} |k(r)r^{\beta+1}|dr$$

- Explicit forms for $k(\cdot) = k_\beta(\cdot)$ (Legendre Polynomial)
  - $k_1(r) = k_2(r) = 3r$
  - $k_3(r) = k_4(r) = \frac{15r}{4}(5 - 7r^3)$
  - $k_5(r) = k_6(r) = \frac{195r}{64}(99r^4 - 126r^2 + 35)$

Both tricks combined "smoothing"

- Smoothened β-smooth mapping f (with unbiased estimate)

$$f_{r,\delta}(x) = \mathbb{E}_r \mathbb{E}_{\|v\|\leq 1} f(x + r\delta v) r k(r)$$

- $\nabla f_{r,\delta}(x) = \mathbb{E}_r \mathbb{E}_{\|u\|=1} \frac{d}{\delta} f(x + r\delta u) k(r) u$

$$\left| f_{r,\delta}(x) - f(x) \right| \leq \frac{M_\beta^\beta}{\beta!} \delta^\beta \mathbb{E}_r |k(r) r^{\beta+1}|$$

$$\left| \nabla f_{r,\delta}(x) - \nabla f(x) \right| \leq \frac{M_\beta^\beta}{\beta-1!} \delta^{\beta-1} \mathbb{E}_r |k(r) r^{\beta+1}|$$

- $E_r |k(r)|^2 \leq 3\beta^3$
  $E_r |k(r)|^2 r^2 \leq 8\beta^2$
  $E_r |k(r) r^{\beta+1}| \leq 2\sqrt{2}\beta$

- $f_{r,\delta}$ is $\frac{\mu}{2}$-strongly-convex if f is μ-strongly convex (and δ small)
  $f_{r,\delta}$ is convex if f convex and $\beta \leq 2$

Two meta-algorithms

Constrained $\mathcal{X}$ compact, one query $f(\xi) + \varepsilon$ per iteration

One Point algorithm

$$x_t = \Pi_{\mathcal{X}}\left(x_{t-1} - \gamma_t \frac{d}{\delta_t}\left[f(x_{t-1} + \delta_t r_t u_t) + \varepsilon_t\right] k(r_t) u_t\right) \qquad \text{(A1)}$$

$r_t \sim \mathcal{U}([-1, 1])$, $u_t \sim \mathcal{U}(\mathbb{S}^d)$
γ_t, δ_t are deterministic sequences

Unconstrained $\mathcal{X} = \mathbb{R}^d$, $f(\xi_t^1)$ and $f(\xi_t^2)$ (independent noises)

Two points algorithm

$$x_t = x_{t-1} - \gamma_t \frac{d}{2\delta_t}\left[f(x_{t-1} + \delta_t r_t u_t) - f(x_{t-1} - \delta_t r_t u_t) + \varepsilon_t\right] k(r_t) u_t \qquad \text{(A2)}$$

μ-Strongly convex + Constrained

- 1-point meta-algo, f is β-smooth for $\beta \geq 2$

$$x_t = \Pi_{\mathcal{X}}\left(x_{t-1} - \gamma_t \frac{d}{\delta_t}\left[f(x_{t-1} + \delta_t r_t u_t) + \varepsilon_t\right] k(r_t) u_t\right)$$

- Choice of parameters
  - $\gamma_t = \frac{1}{\mu t}$ (classic choice for μ-strongly)
  - $\delta_t = \left(2\frac{d^2\beta^2\beta!}{t\mu M_\beta^\beta}\right)^{\frac{1}{\beta+2}}$ (remark $\beta!^{1/\beta} \sim \beta/e$)
- Output. Averaging $x_T^\star = \frac{1}{T}\sum_{t=1}^{T} x_t$
- Convergence guarantee (leading term)

$$\mathbb{E}f(x_T^\star) - f^\star \leq 12\beta^2\left(\frac{2d^2M_\beta^2}{\mu T}\right)^{\frac{\beta}{\beta+2}}((M_0 + M_1)^2 + \sigma^2 + 1)$$

Sketch of proof - 6 steps

1) Definition of x_t by algo

$$\|x_t - x\|^2 \leq \|x_{t-1} - x\|^2 - 2\gamma_t \frac{d}{\delta_t}\left[f(x_{t-1} + \delta_t r_t u_t) + \varepsilon_t\right] k(r_t) u_t^\top (x_{t-1} - x) + 2|k(r_t)|^2 \frac{\gamma_t^2 d^2}{\delta^2}\left[|f(x_{t-1} + \delta_t r_t u_t)|^2 + |\varepsilon_t|^2\right]$$

2) $\mu/2$-strong convexity of f_δ

$$\mathbb{E}\|x_t - x\|^2 \leq \mathbb{E}\|x_{t-1} - x\|^2 - 2\gamma_t \left(\mathbb{E} f_{\delta_t}(x_{t-1}) - \mathbb{E} f_{\delta_t}(x) + \frac{\mu}{4}\mathbb{E}\|x_{t-1} - x\|^2\right) + \frac{\gamma_t^2 d^2}{\delta_t^2} C$$

3) Rearranging

$$\mathbb{E} f_{\delta_t}(x_{t-1}) - \mathbb{E} f_{\delta_t}(x) \leq \mathbb{E}\|x_{t-1} - x\|^2 \left(\frac{1}{2\gamma_t} - \frac{\mu}{2}\right) - \mathbb{E}\|x_t - x\|^2 \frac{1}{2\gamma_t} + \frac{\gamma_t d^2}{\delta_t^2} C$$

4) The choice of $\gamma_t = 1/\mu t$

$$\mathbb{E} f_{\delta_t}(x_{t-1}) - \mathbb{E} f_{\delta_t}(x) \leq \frac{(t-1)\mu}{2}\mathbb{E}\|x_{t-1} - x\|^2 - \frac{t\mu}{2}\mathbb{E}\|x_t - x\|^2 + \frac{d^2}{t\mu\delta_t^2} C$$

5) Summing over t and averaging

$$\frac{1}{T}\sum_{t=1}^T \mathbb{E}\left[f_{\delta_t}(x_{t-1}) - f_{\delta_t}(x)\right] + \frac{\mu}{2}\mathbb{E}\|x_T - x\|^2 \leq \frac{d^2}{\mu}\frac{1}{T}\sum_{t=1}^T \frac{1}{t\delta_t^2} C$$

6) Plugging back f and balancing

$$\mathbb{E} f(x_T^\star) - f^\star \lesssim \frac{d^2}{\mu}\frac{1}{T}\sum_{t=1}^T \frac{1}{t\delta_t^2} + \frac{1}{T}\sum_{t=1}^T \delta_t^\beta \lesssim \frac{1}{T}\sum_{t=1}^T \left(\frac{d^2}{\mu t}\right)^{\frac{\beta}{\beta+2}} \simeq \left(\frac{d^2}{\mu T}\right)^{\frac{\beta}{\beta+2}}$$

Strongly convex + Unconstrained

- 2-point meta-algo, f is β-smooth for $\beta \geq 2$

$$x_t = x_{t-1} - \gamma_t \frac{d}{2\delta_t}\left[f(x_{t-1} + \delta_t r_t u_t) - f(x_{t-1} - \delta_t r_t u_t) + \varepsilon_t\right] k(r_t) u_t$$

- Choice of parameters
  - $\gamma_t = \frac{2}{\mu(t+1)}$ (classic choice for μ-strongly & 2-smooth)
  - $\delta_t = \delta = \left(2\frac{d^2\beta^2\beta!}{t\mu M_\beta^\beta}\right)^{\frac{1}{\beta+2}}$ (constant step size)
- Output. Uniform averaging $x_T^\star = \frac{1}{T}\sum_{t=1}^T x_t$
- Convergence guarantee (leading term)

$$\mathbb{E}f(x_T^\star) - f^\star \leq 27\beta\left(\frac{2d^2M_\beta^2}{\mu T}\right)^{\frac{\beta}{\beta+2}}(\sigma^2 + 2)$$

Remarks on strongly convex

- Constrained & unconstrained, same rates $\left(\frac{2d^2M_\beta^2}{\mu T}\right)^{\frac{\beta}{\beta+2}}$
  - In the proof, Step 5 yields

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[f_{\delta_t}(x_{t-1})-f_{\delta_t}(x)\right]+\frac{\mu}{2}\mathbb{E}\|x_T-x\|^2\lesssim\frac{d^2}{\mu}\frac{1}{T}\sum_{t=1}^{T}\frac{1}{t\delta_t^2}$$

- Using this for $\frac{\mu}{2}\mathbb{E}\|x_T-x\|^2$, improve rates into

$$\mathbb{E}\|x_T-x^\star\|^2\leq 2\mathbb{E}\|x_T-x_\delta^\star\|^2+2\mathbb{E}\|x_\delta^\star-x^\star\|^2\lesssim\frac{1}{\mu^2}\left(\frac{d^2}{T}\right)^{\frac{\beta-1}{\beta}}$$

Convex + constrained

- 1-point meta-algo, f is β-smooth for $\beta > 2$

$$x_t = \Pi_{\mathcal{X}}\left(x_{t-1} - \gamma_t \frac{d}{\delta_t}\left[f(x_{t-1} + \delta_t r_t u_t) + \varepsilon_t\right] k(r_t) u_t\right)$$

- Choice of parameters
  - $\gamma_t = \frac{\delta_t R}{\beta^{3/2} d\sqrt{t}}$ (classic choice for non-strongly)
  - $\delta_t = \left(\frac{dR\sqrt{\beta}\beta!}{\sqrt{t} M_\beta^\beta}\right)^{\frac{1}{\beta}}$
- Output. Averaging $x_T^\star = \frac{1}{T}\sum_{t=1}^{T} x_t$
- Convergence guarantee (leading term)

$$\mathbb{E} f(x_T^\star) - f^\star \leq 19\sqrt{\beta}\beta^2 \left(\sqrt{\frac{d^2 M_\beta^2 R^2}{T}}\right)^{\frac{\beta-1}{\beta}} ((M_0 + M_1)^2 + \sigma^2 + 1)$$

Remarks on convex, $\beta = 2$

- Rates for $\beta > 2$, $\mathbb{E}f(x_T^\star) - f^\star \lesssim \left(\sqrt{\frac{d^2}{T}}\right)^{\frac{\beta-1}{\beta}}$
- For $\beta = 2$, we can get $\mathbb{E}f(x_T^\star) - f^\star \lesssim \left(\sqrt{\frac{d^2}{T}}\right)^{\frac{\beta}{\beta+1}} = \left(\frac{d^2}{T}\right)^{\frac{1}{3}}$

The mapping f_δ is convex for $\beta = 2$, not necessarily $\beta > 2$.

- The first lines of proof are modified
  - Instead stoch. gradient descent of f_δ + **approximation** of f by f_δ
  - **Approximation** of $\nabla f(x)$ by $\nabla f_\delta(x)$ + stoch. grad descent of f.
- Loose 1 in the regularity number ($\beta \rightsquigarrow \beta - 1$)

Convex + Unconstrained

- 2-point meta-algo, f is β-smooth for $\beta \geq 2$

$$x_t = x_{t-1} - \gamma_t \frac{d}{2\delta_t}\left[f(x_{t-1} + \delta_t r_t u_t) - f(x_{t-1} - \delta_t r_t u_t) + \varepsilon_t\right] k(r_t) u_t$$

- Choice of parameters
  - $\gamma_t = \frac{\delta}{\beta^2 d M_2 \sqrt{T}}$ (classic choice for μ-strongly & 2-smooth)
  - $\delta_t = \delta = \left(\frac{d\beta\beta!}{\sqrt{t} M_\beta^\beta M_2}\right)^{\frac{1}{\beta}}$ (constant step size)
- Output. Non-uniform averaging $x_T^\star = \frac{2}{T(T+1)} \sum_{t=1}^{T} t x_t$
- Convergence guarantee (leading term)

$$\mathbb{E} f(x_T^\star) - f^\star \leq \beta \left(\sqrt{\frac{2d^2 M_\beta^2}{T M_2^2}}\right)^{\frac{\beta-1}{\beta}} (96 M_2^2 \|x_0 - x^\star\|^2 + \sigma^2 + 20)$$

Sum up - Objectives

Principal rates (without smoothness for the 1st column)

	1st	noisy	0th	0/noise	Our results
cvx	$\sqrt{\frac{1}{T}}$	$\sqrt{\frac{d}{T}}$	$\sqrt{\frac{d}{T}}$	$\sqrt{\frac{d^2}{T}}$?	$\left(\sqrt{\frac{d^2}{T}}\right)^{\frac{\beta-1}{\beta}}$ & $\left(\sqrt{\frac{d^2}{T}}\right)^{\frac{\beta}{\beta+1}}$, $\beta \leq 2$
μ-strg	$\frac{1}{\mu T}$	$\frac{d}{\mu T}$	$\frac{d}{\mu T}$	$\frac{d^2}{\mu T}$?	$\left(\frac{d^2}{\mu T}\right)^{\frac{\beta}{\beta+2}}$ or $\frac{1}{\mu^2}\left(\frac{d^2}{T}\right)^{\frac{\beta+1}{\beta}}$

Existing results

- Minimax speed (not strg) $\frac{\mathrm{poly}(d)}{\sqrt{T}}$ [Bubeck-Eldan],[Rakhlin et al.]
- Strongly AND smooth, algo $\sqrt{\frac{d^2}{\mu T}}$[Hazan, Levy]
- Strongly OR smooth, algo $T^{-1/3}$ [Agarwal et al][Saha,Tewari])
- Only Convex $T^{-1/4}$ [Flaxman et al][Kleinberg]
- Strongly and β-smooth, $T^{-\frac{\beta+1}{\beta}}$ [Polyak,Tsybakov]

Online Learning & Bandits

- Remember Step 5 of the proof in off-line

5) Summing over t and averaging

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[f(x_{t-1})-f(x)\big]+\frac{\mu}{2}\mathbb{E}\|x_T-x\|^2\lesssim\frac{d^2}{\mu}\frac{1}{T}\sum_{t=1}^{T}\frac{1}{t\delta_t^2}+\frac{1}{T}\sum_{t=1}^{T}\delta_t^{\beta}$$

The first term is the regret !

- **All results with uniform averaging in off-line holds on-line**
- Additional log(T) factor for unconstrained strongly convex
- In bandit learning, $x_t^\star$ must be equal to x_t (the query point),
  - Theoretical results $\mathrm{poly}(d)/\sqrt{n}$ for convex mappings
  - Algo: $\sqrt{T}$ for $\beta = 2$ **and** strongly convex
  - Algo: $T^{-1/4}$ for convex, $T^{-1/3}$ for smooth **or** strongly convex